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COMPUTATION OF EFFECTIVE PLASTICITY CHARACTERISTICS OF INHOMOGENEOUS MEDIA

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Macroscopic mechanical characteristics of a composite material, representing a mixture of inclusions and host, are determined by the mechanical properties of the phases and its geometric configuration. We define the composite configuration by that uniform distribution of the spherical inclusions in the host so that the characteristic function $\varkappa$ equals one at the inclusions and zero at the host and is statistically homogeneous and isotropic. With respect to the mechanical properties of the phases, we limit ourselves to the condition that the plastic properties of the inclusions be higher than the plastic properties of the host. Hence, the host can be considered ideally elastic in a definite deformation range, and the inclusions ideally elastic–plastic Both phases are interconnected such that slip of the inclusions in the host is excluded.

1. The materials of the host and the inclusions are considered isotropic and Hooke's law in the phases is written in the form

$$\sigma_{ij} = 2\mu_\alpha (e_{ij} - e_{ij}^p) + \delta_{ij}\lambda_\alpha e_{kk},$$

where μ_α, λ_α are the Lamé parameters, σ_{ij}, e_{ij}, e_{ij}^p are components of the stress, the total and plastic deformation tensors, and $\alpha = 1$ corresponds to the host and $\alpha = 2$ to the inclusion. The plastic deformations satisfy the incompressibility condition $e_{kk}^p = 0$. The plastic properties of the inclusions are determined by the Mises plasticity condition $s_{ij}s_{ij} = k^2$, where s_{ij} and k are the deviator components of the stress tensor and the plasticity limit of the inclusions, respectively.

An investigation of the extremum [1] of the function

$$L = \frac{1}{V}\left\{\int\limits_V \left[D(\varepsilon_{ij}^p) + \frac{1}{2} W(e_{ij} - e_{ij}^p, \varepsilon_{ij} - \varepsilon_{ij}^p)\right] dV - \int\limits_S (p_i v_i + q_i u_i)\, dS\right\} \quad (1.1)$$

determines the properties of the inhomogeneous medium. Here $D(\varepsilon_{ij}^p) = k(x)\sqrt{\varepsilon_{ij}^p \varepsilon_{ij}^p}$ is the dissipative function for the selected plasticity condition [2];

$$\frac{1}{2} W(e_{ij} - e_{ij}^p;\ \varepsilon_{ij} - \varepsilon_{ij}^p) = 2\mu(x)(e_{ij} - e_{ij}^p)(\varepsilon_{ij} - \varepsilon_{ij}^p) + \lambda(x) e_{kk}\varepsilon_{kk}$$

is the rate of change of the elastic energy, ε_{ij}, ε_{ij}^p are the components of the total and plastic strain rate tensors, u_i, v_i are the displacements and the velocities, p_i, q_i are the loads and their velocities on the surface. The total volume V is a simply connected domain. The random stress, strain, and their velocity fields

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are assumed ergodic so that their mathematical expectations, agreeing with the means in the volume of values, are independent of the coordinates. Taking the average with respect to V, the volume of inclusions V_i is denoted by

$$\langle(\ldots)\rangle = \frac{1}{V}\int_V (\ldots)\,dV, \quad \langle(\ldots)\rangle_i = \frac{1}{V_i}\int_{V_i} (\ldots)\,dV.$$

The quantities $\langle e_{ij}\rangle$, $\langle \varepsilon_{ij}\rangle$, $\langle \sigma_{ij}\rangle$ and the residual strains e^*_{ij} are considered given, and the k(x), μ(x), λ(x) form homogeneous ergodic random fields. The fields of the displacement fluctuations u'_i and their velocities v'_i are sought in the class of statistically homogeneous continuous functions.

With respect to taking the average of the field magnitudes over the inclusions, it is assumed that

$$\langle e_{ij}\varepsilon_{kl}\rangle_i = \langle e_{ij}\rangle_i \langle \varepsilon_{kl}\rangle_i. \tag{1.2}$$

Such an assumption is completely justified for low inclusion concentrations in elasticity theory since it corresponds to a homogeneous strain state of each spherical inclusion [3]. For higher concentrations of inclusions, this assumption should be satisfied because of the equilibrated mutual influence of the spheres, which are distributed statistically homogeneously in the volume V.

The condition on the boundary S of the volume V has the form

$$p_i = \langle \sigma_{ij}\rangle n_j, \; g_i = \langle \dot{\sigma}_{ij}\rangle n_j, \tag{1.3}$$

where n_j is the vector of the unit normal to S and $\dot\sigma_{ij}$ is the velocity of the stress field.

Dividing the volume V into $V - V_i$ and V_i, we obtain by using (1.1)-(1.3)

$$\begin{aligned} L = {} & k\sqrt{\langle \varepsilon^p_{ij}\rangle\langle \varepsilon^p_{ij}\rangle} + 2\mu_1\langle e_{ij}\varepsilon_{ij}\rangle + \lambda_1\langle e_{kk}\varepsilon_{kk}\rangle + 2\Delta\mu c\langle e_{ij}\rangle_i\langle \varepsilon_{ij}\rangle_i \\ & + \Delta\lambda c\langle e_{kk}\rangle_i\langle \varepsilon_{kk}\rangle_i + 2\mu_2 c^{-1}\langle e^p_{ij}\rangle\langle \varepsilon^p_{ij}\rangle - 2\mu_2\big(\langle e_{ij}\rangle_i\langle \varepsilon^p_{ij}\rangle \\ & + \langle e^p_{ij}\rangle\langle \varepsilon_{ij}\rangle_i\big) - \langle \sigma_{ij}\rangle\langle \varepsilon_{ij}\rangle - \langle \dot\sigma_{ij}\rangle\langle e_{ij}\rangle, \end{aligned} \tag{1.4}$$

where the obvious relationship $\langle e^p_{ij}\rangle = c\langle e^p_{ij}\rangle_i$ has been used and $c = V_i V^{-1}$ is the volume concentration of the inclusions $\Delta\mu = \mu_2 - \mu_1$, $\Delta\lambda = \lambda_2 - \lambda_1$.

The disappearance of the first variation of (1.4) at independent variations of the fluctuations $\delta u'_i$, $\delta v'_i$ results in two systems of equations

$$2\mu_1 e'_{ij,j} + \lambda_1 e'_{kk,i} = a_{ij}\varkappa'_{,j}, \quad 2\mu_1\varepsilon'_{ij,j} + \lambda_1\varepsilon'_{kk,i} = b_{ij}\varkappa'_{,j},$$

whose right side depends on the strain state of the inclusions and the mathematical expectation of the plastic deformation.

Here

$$a_{ij} = 2\mu_2 c^{-1}\langle e^p_{ij}\rangle - 2\Delta\mu\langle e_{ij}\rangle_i - \delta_{ij}\Delta\lambda\langle e_{kk}\rangle_i;$$
$$b_{ij} = 2\mu_2 c^{-1}\langle \varepsilon^p_{ij}\rangle - 2\Delta\mu\langle \varepsilon_{ij}\rangle_i - \delta_{ij}\Delta\lambda\langle \varepsilon_{kk}\rangle_i.$$

The solution is written by using the Green's tensor $U^i_k(x, \xi)$ [4]

$$e'_{ij}(x) = a_{kl}\int_V G_{i(k,l)j}(x, \xi)\,\varkappa'(\xi)\,dV,$$
$$\varepsilon'_{ij}(x) = b_{kl}\int_V G_{i(k,l)j}(x, \xi)\,\varkappa'(\xi)\,dV,$$

where $2G_{ik,lj}(x, \xi) = [\partial U^i_k(x, \xi)/(\partial\xi_l\partial x_j) + \partial U^j_k(x, \xi)/(\partial\xi_l\partial x_i)]$ and the parentheses denote symmetrization with respect to the subscripts.

2. To determine the effective characteristics of a composite, it is necessary to express the functional (1.4) in terms of the field magnitudes averaged with respect to the volume V. This can be done by using the obvious equalities

$$\langle e_{ij}\rangle_1 = \langle e_{ij}\rangle + c^{-1}\langle \varkappa' e'_{ij}\rangle, \quad \langle \varepsilon_{ij}\rangle_1 = \langle \varepsilon_{ij}\rangle + c^{-1}\langle \varkappa' \varepsilon'_{ij}\rangle. \tag{2.1}$$

For statistically homogeneous and isotropic functions $\varkappa$ and $\varkappa'$ the binary correlation function is

$$\langle \varkappa'(x)\varkappa'(\xi)\rangle = f(r),$$

where $f(r)$ is some function of the distance $r^2 = (x_i - \xi_i)(x_i - \xi_i)/R^2$, and R is the radius of the inclusions.

Then

$$\begin{aligned}
\langle \varkappa' e'_{ij}\rangle &= \frac{c(1-c)}{30\mu_1(1-\nu_1)}[2(4-5\nu_1)a_{ij} - \delta_{ij}a_{kk}],\\
\langle \varkappa' \varepsilon'_{ij}\rangle &= \frac{c(1-c)}{30\mu_1(1-\nu_1)}[2(4-5\nu_1)b_{ij} - \delta_{ij}b_{kk}],\\
2\mu_1\langle e'_{ij}\varepsilon'_{ij}\rangle + \lambda_1\langle e'_{kk}\varepsilon'_{kk}\rangle &= \frac{c(1-c)}{30\mu_1(1-\nu_1)}[2(4-5\nu_1)a_{ij}b_{ij} - a_{kk}b_{kk}]
\end{aligned} \tag{2.2}$$

(ν_1 is Poisson's ratio in the host).

Substituting the first of formulas (2.2) into (2.1), taking account of the expression for a_{ij}, we obtain the value of strain averaged over the inclusions in terms of the total strain

$$\langle e_{ij}\rangle_1 = A\langle e_{ij}\rangle + \delta_{ij}B\langle e_{kk}\rangle + C\langle e''_{ij}\rangle, \tag{2.3}$$

(an analogous expression is valid for $\langle \varepsilon_{ij}\rangle_1$) where

$$\begin{aligned}
A &= \frac{15\mu_1(1-\nu_1)}{15\mu_1(1-\nu_1) + 2(1-c)\Delta\mu(4-5\nu_1)};\\
B &= A\frac{(1-c)[2\Delta\mu - 5\Delta\lambda(1-2\nu_1)]}{5[6\mu_1(1-\nu_1) + (1-c)(1-2\nu_1)(2\Delta\mu + 3\Delta\lambda)]};\\
C &= A\frac{2\mu_2(1-c)(4-5\nu_1)}{15\mu_1 c(1-\nu_1)}.
\end{aligned}$$

By using (2.2) and (2.3), the functional (1.4) is expressed in terms of the magnitudes of the fields averaged just with respect to the total volume V. Hence, the angular brackets are henceforth omitted.

The stationarity condition for such a functional in ε_{ij} yields the effective Hooke's law of an inhomogeneous medium. The deviator part of this law is

$$s_{ij} = 2\mu^*(e^d_{ij} - \overset{*}{e}_{ij}). \tag{2.4}$$

Here

$$\mu^* = \mu_1\left[1 + \frac{15(1-\nu_1)c(m-1)}{15(1-\nu_1) + 2(1-c)(m-1)(4-5\nu_1)}\right]$$

is the effective shear modulus of the composite, $m = \mu_2/\mu_1$; and e^d_{ij} is the strain tensor deviator. The volume part of Hooke's law is written in the form

$$\sigma_{kk} = 3K^* e_{kk},$$

where

$$K^* = \frac{K_1(3K_2 + 4\mu_1) + 4\mu_1(K_2 - K_1)c}{(3K_2 + 4\mu_1) + 3c(K_1 - K_2)}$$

is the effective elastic modulus of multilateral tension (compression), and K_1, K_2 are the volume phase moduli. The formulas for μ^* and K^* agree with the expressions in [5, 6].

The elastic moduli of a composite fabricated from P-47 polystyrene filled with glass microspheres were investigated experimentally in [7]. The formulas for μ^* and K^* displayed quite good agreement with the experimental results.

Taking account of (2.4), the condition for stationarity of the functional in ε_{ij}^{p} yields the plasticity condition for a composite material

$$k^{*2} = (s_{ij} - Ne_{ij}^*)(s_{ij} - Ne_{ij}^*),$$

where

$$k^* = k\left[\frac{1}{m} + \frac{2}{15}\frac{4-5\nu_1}{1-\nu_1}(1-c)\left(1-\frac{1}{m}\right) + c\left(1-\frac{1}{m}\right)\right]$$

is the effective limit of plasticity of the medium;

$$N = 2\mu^*\left[\frac{k^*}{kc}\frac{(7-5\nu_1)+2c(4-5\nu_1)}{15(1-\nu_1)} - 1\right]$$

is the coefficient of linear hardening characterizing the displacement of the flow surface under loading. The coefficient of linear hardening is a fractional-linear function of the concentration and varies between infinity and zero as the concentration changes from zero to one. The plasticity limit k^* is a linear function of the concentration, that equals K for $c = 1$. For $c = 0$

$$k^* = k\left[\frac{1}{m} + \frac{2}{15}\frac{4-5\nu_1}{1-\nu_1}\left(1-\frac{1}{m}\right)\right],$$

this quantity is the exact upper bound of all possible $k^* = k^*(c)$ which characterize the initial flow surface of the composite.

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